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The Gibbs Phenomenon for L^1_{loc} Kernels*

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For a large class of $L^1_{\text{loc}}(\mathbb{R}^n)$ kernels we give sharp estimates of the Gibbs phenomenon, by reducing the problem to the one-dimensional case. © 1999 Academic Press

1. INTRODUCTION

In 1910, Hermann Weil in two papers [4, 5] studied, among other questions, the Gibbs phenomenon for two-dimensional spherical harmonics expansions and, in order to give numerical estimates of the phenomenon, reduced the problem to the case of a one-dimensional Fourier series.

Recently, L. Colzani and M. Vignati obtained a result in the same spirit for the Gibbs phenomenon connected to radial kernels for multiple Fourier integrals in $\mathbb{R}^n$. They considered a domain C in $\mathbb{R}^n$ whose boundary ∂C is a smooth simple closed surface and a function $f\colon \mathbb{R}^n \to \mathbb{R}$ defined as

$$f(P) = \begin{cases} f_C(P) & \text{if} \quad P \in C \\ 0 & \text{if} \quad P \in \mathbb{R}^n \setminus C, \end{cases}$$

where $f_C = C \to \mathbb{R}$ is a continuous function.

Then, if G is a radial $L^1(\mathbb{R}^n)$ function and $G_\sigma(x) = (1/\sigma^n)\, G(x/\sigma)$, they proved that, with some technical hypotheses on G, the behaviour of $G_\sigma * f$ in a neighbourhood of a point $x_0 \in \partial C$ on the exterior normal v to ∂C in x_0 is the same as the behaviour of $g_\sigma * \tilde{f}$, where $\tilde{f}$ is the restriction of f to v and g_σ is a suitable one-dimensional kernel closely related to G_σ [1, Theorem 1].

Moreover, they extended the result to Bochner–Riesz means S^α of every order $\alpha \geqslant 0$ in $\mathbb{R}^2$ and they observed that in $\mathbb{R}^n$ $(n > 2)$ if $\alpha \leqslant (n-3)/2$ it is not possible to obtain a similar result.

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By a different technique, the authors in [2] proved that for this kind of problem and for every $L^1(\mathbb{R}^n)$ kernel, the study of the Gibbs phenomenon can be reduced to the one-dimensional case. Moreover, the estimate still holds when x moves to x_0 in any bilateral cone not tangential to ∂C and having as axis the normal v to ∂C passing through x_0.

In this paper, we consider a family of $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ kernels with some property of "conditional integrability" in $\mathbb{R}^n$, also with the aim of giving an evaluation of the Gibbs phenomenon for the Bochner–Riesz means S^α with $\alpha > (n-3)/2$.

We prove that, for this kind of kernels and for smooth functions f, $G_\sigma * f \to f$ uniformly on every compact set of $\mathbb{R}^n$ disjoint from ∂C and that, under suitable conditions on ∂C, again the Gibbs phenomenon can be evaluated as in the $L^1(\mathbb{R}^n)$ case.

2. THE RESULTS

Let $x = (x_1, ..., x_n) \in \mathbb{R}^n$, $n \geqslant 1$. For convenience, we consider the two following norms:

$$|x| = \sup_j |x_j|, \qquad \|x\| = \left\{ \sum_{j=1}^n x_j^2 \right\}^{1/2}.$$

Let

$$Q(y, r) = \{x \in \mathbb{R}^n : |x - y| \leqslant r\},$$

$$\Gamma_\lambda = \left\{ x \in \mathbb{R}^n : x_1^2 \geqslant \lambda \sum_{j=2}^n x_j^2 \right\}, \qquad \lambda \geqslant 0.$$

In the sequel, we consider functions $G\colon \mathbb{R}^n \to \mathbb{R}$, $G \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ with the following property. If Σ is the family of the n-cells

$$S = \{x \in \mathbb{R}^n : a_j < x_j < b_j;\, a_j, b_j \in \mathbb{R},\, a_j < b_j,\, j = 1, ..., n\},$$

for every $\varepsilon > 0$ there exists $a_0 = a_0(\varepsilon)$ such that

$$\left| \int_S G(x)\, dx \right| < \varepsilon \qquad \forall S \in \Sigma, \quad S \cap Q(0, a_0(\varepsilon)) = \varnothing. \tag{2.1}$$

We preliminary observe that for these G obviously for every $S \in \Sigma$, $0 \in S$, there exists $\lim_{r \to \infty} \int_{rS} G(x)\, dx = a < +\infty$, and such limit does not depend on S. So we can always suppose $a = 1$.

Moreover, for every $s \in \mathbb{R}$ there exists

$$\lim_{r \to \infty} \int_{\substack{s < x_1 < r \\ |x_i| < r,\, i = 2, \ldots, n}} G(x)\, dx < +\infty$$

which we will denote by $\int_{x_1 > s} G(x)\, dx$.

As usual for every $\sigma > 0$ we set

$$G_\sigma(x) = \frac{1}{\sigma^n}\, G\left(\frac{x}{\sigma}\right).$$

Finally, let $C \subset \mathbb{R}^n$ be a compact set with a simple surface as boundary, smooth enough at least in a neighbourhood of $x = 0$, for which the positive part of x_1 axis is the exterior normal at the origin. Moreover, we suppose that for every $x \in \partial C$ there exists $S \in \Sigma$, $S \ni x$ such that the intersections of $C \cap S$ with the straight lines parallel to the coordinate axes are connected.

Let χ the characteristic function of C. Then we have the following

THEOREM 1. *Suppose that for some constant c and for every ε sufficiently small the following condition holds,*

$$\left| \int_{S \cap (x - \sigma C)} G(y)\, dy \right| < c\varepsilon \tag{2.2}$$

for every $S \in \Sigma$, $S \cap Q(0, a_0(\varepsilon)) = \varnothing$, $x \in \mathbb{R}^n$, $\sigma > 0$. *Let* $f\colon \mathbb{R}^n \to \mathbb{R}$ *be a function vanishing in* $\mathbb{R}^n \backslash C$ *and smooth enough in C. Then* $G_\sigma * f \to f$ *as* $\sigma \to 0$ *uniformly in every compact subset of* $\mathbb{R}^n \backslash \partial C$. *Moreover*

$$G_\sigma * f(t) = \chi(t)\{f(t) - f(0)\} + f(0) \int_{x_1 > t_1/\sigma} G(x)\, dx + \eta(t, \sigma), \tag{2.3}$$

where $t = (t_1, \ldots, t_n)$ *and* $\eta(t, \sigma) \to 0$ *if* $\sigma \to 0$ *uniformly with respect to t in a neighbourhood of* $t = 0$ *in* Γ_λ.

The result can be applied to the following (non-radial) oscillating kernels: $G(x) = x_j e^{i\,\|x\|^2} / \|x\|^\alpha$, $x \in \mathbb{R}^n$, $\alpha \geqslant n$. Moreover also suitable partial derivatives of the Bochner–Riesz kernels fall within the scope of the theorem.

Remarks. (1) If $n = 1$, formula (2.3) shows that $f(0) \int_{x_1 < t_1/\sigma} G(x)\, dx$ and $f(0) \int_{x_1 > t_1/\sigma} \cdot G(x)\, dx$ give the main part of the oscillation of $G_\sigma * f(t) - f(t)$ in a neighbourhood of $t = 0$, respectively for $t < 0$ and $t > 0$

(the Gibbs phenomenon). If $n>0$, setting $g(x_1)=\int_{\mathbb{R}^{n-1}} G(x_1,...,x_n)\,dx_2\cdots dx_n$, (2.3) shows that

$$G_\sigma * f(t)-f(t)=g_\sigma * \tilde{f}(t_1)-f(t_1)+\eta(t,\sigma),$$

where $\tilde{f}$ is the restriction of f to the x_1 axis.

(2) Theorem 1 extends Theorem 2 of [2] to functions G no necessarily $L^1(\mathbb{R}^n)$, hence, in order to have the uniform convergence of $\eta(t,\sigma)$ to zero, we have to restrict ourselves to a cone Γ_λ. (See the remark in [2] after Theorem 1).

(3) With the same argument as in the proof of Theorem 3 in [2], it is possible to obtain a similar result for compactly supported functions not necessarily vanishing in the complement of C.

With some work, Theorem 1 can be adapted to the radial situation, in the following way.

Let $G\colon \mathbb{R}^n\to\mathbb{R}$ be a radial function $G(x)=\tilde{G}(\|x\|)$, $G\in L^1_{\text{loc}}(\mathbb{R}^n)$ and suppose that the following limit does exist,

$$\lim_{r\to\infty}\int_0^r \rho^{n-1}\tilde{G}(\rho)\,d\rho=\int_0^{\to+\infty}\rho^{n-1}\tilde{G}(\rho)\,d\rho=\frac{1}{m_n}, \tag{2.4}$$

where $\rho=\|x\|$ and m_n is the surface measure of the unit sphere B_n in $\mathbb{R}^n$.

Then we have

THEOREM 2. *With the previous hypotheses on G radial, if (2.2) holds and $f\colon \mathbb{R}^n\to\mathbb{R}$ is a function vanishing in $\mathbb{R}^n\backslash C$ and smooth enough in C, then $G_\sigma * f\to f$ as $\sigma\to 0$ uniformly in every compact subset of $\mathbb{R}^n\backslash\partial C$. Moreover*

$$G_\sigma * f(t)=\chi(t)\{f(t)-f(0)\}+m_{n-1}f(0)$$
$$\times\int_{t_1/\sigma}^{\to+\infty}dx_1\int_0^{\to+\infty}r^{n-2}\,\tilde{G}\left(\sqrt{x_1^2+r^2}\right)dr+\eta(t,\sigma), \tag{2.5}$$

where $r=\{\sum_{j=2}^n x_j^2\}^{1/2}$ and $\eta(t,\sigma)\to 0$ if $\sigma\to 0$ uniformly with respect to t in a neighbourhood of $t=0$ in Γ_λ.

Remarks. (1) This result is new also for $L^1(\mathbb{R}^n)$ kernels, but in this case is an easy corollary of Theorem 2 of [2].

(2) Theorem 2 can be applied, e.g., to the Bochner–Riesz means also for some indices α below to the critical index $\bar{\alpha}=(n-1)/2$ (Section 6).

3. A TECHNICAL LEMMA

Let H: $\mathbb{R}^n \to \mathbb{R}$, $H \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that

$$\|H\| = \sup_{S \in \Sigma} \left| \int_S H(x)\, dx \right| < +\infty.$$

Let $K = \{x \in \mathbb{R}^n : 0 \leqslant x_i \leqslant r_i,\ r_i > 0,\ i = 1, ..., n\}$ and let D be a compact subset of K with the property

$$\bar{x} \in D \Rightarrow x \in D \qquad \forall x\colon 0 \leqslant x_i \leqslant \overline{x_i}, \quad i = 1, ..., n.$$

LEMMA 1. *Let f: $\mathbb{R}^n \to \mathbb{R}$, $f \in \mathscr{C}^{(n)}(D)$ vanishing in $\mathbb{R}^n \backslash D$. Suppose that for some constant c*

$$\left| \int_{S \cap (x - \sigma D)} H(y)\, dy \right| < c\, \|H\| \tag{3.1}$$

for every $S \in \Sigma$, $x \in \mathbb{R}^n$, $\sigma > 0$. Then there exists a constant $M = M(f)$ such that for every $t \in \mathbb{R}^n$ and for every $\sigma > 0$ we have

$$|H_\sigma * f(t)| \leqslant cM\, \|H\|. \tag{3.2}$$

Proof. Preliminary we observe that if for every $x = (x_1, ..., x_n) \in D$

$$f(x) = \int_0^{x_1} \cdots \int_0^{x_j} \varphi(y_1, ..., y_j)\, dy_1, ..., dy_j$$

with $1 \leqslant j \leqslant n$ and $\varphi \in L^1(\mathbb{R}^j)$, then

$$|H_\sigma * f(t)| \leqslant c\, \|\varphi\|_{L^1(\mathbb{R}^j)}\, \|H\|. \tag{3.3}$$

Indeed

$$\begin{aligned} H_\sigma * f(t) &= \frac{1}{\sigma^n} \int_{\mathbb{R}^n} H\left(\frac{x}{\sigma}\right) f(t - x)\, dx \\ &= \frac{1}{\sigma^n} \int_D f(x)\, H\left(\frac{t - x}{\sigma}\right) dx \\ &= \frac{1}{\sigma^n} \int_D dx \int_{E(x)} \varphi(y)\, H\left(\frac{t - x}{\sigma}\right) dy, \end{aligned}$$

where $E(x) = \{ y \in \mathbb{R}^j : 0 \leqslant y_i \leqslant x_i,\ i = 1, ..., j\}$.

Then, changing variables we obtain

$$H_\sigma * f(t) = \int_{D \cap \mathbb{R}^j} \varphi(y)\, dy \int_{\Omega(y)} H(x)\, dx,$$

where $\Omega(y) = \{x \in (t - D)/\sigma : (t - (y_i + r_i))/\sigma \leqslant x_i \leqslant (t_i - y_i)/\sigma \quad (i = 1, ..., j)$ and $(t - r_j)/\sigma \leqslant x_i \leqslant t/\sigma \ (i = j+1, ..., n)\}$.

This proves (3.3) by (3.1).

Now we prove the lemma by induction on the number of the effective variables of f.

If $f(x) = f(0, ..., 0, x_j, 0, ..., 0,)$ we have

$$f(x) = \int_0^{x_j} f'_{x_j}(0, ..., 0, y_j, 0, ..., 0)\, dy_j + f(0, ..., 0)$$

and the lemma in this case follows by (3.3).

If f effectively depends on $n > 1$ variables, an easy computation shows that

$$\begin{aligned} f(x_1, ..., x_n) &= \sum_{j=1}^{n} \int_0^{x_1} \cdots \int_0^{x_j} f^{(j)}_{x_1, ..., x_j}(y_1, ..., y_j, 0, x_{j+2}, ..., x_n) \\ &\quad \times dy_1, ..., dy_j + f(0, x_2, ..., x_n) \\ &= \int_0^{x_1} \cdots \int_0^{x_n} f^{(n)}_{x_1, ..., x_n}(y_1, ..., y_n)\, dy_1, ..., dy_n \\ &\quad + \sum_{j=1}^{n} F_j(x_1, ..., x_j, 0, x_{j+2}, ..., x_n), \end{aligned}$$

where $F_j \in C^{(n)}(D)$. Then the result follows applying (3.3) to the first term and the hypothesis of induction to the functions F_j, which effectively depend on $n-1$ variables.

Remark. If the hypotheses of the lemma hold in a domain D' obtained from D by translations and symmetries with respect to the coordinate hyperplanes, then (3.2) still holds; then, of course, the same is true for a compact set which is a finite union of such domains D'.

4. PROOF OF THEOREM 1

In order to prove the theorem we have to split the kernel in two parts. The first one is compactly supported and its behaviour is controlled in the cone Γ_λ by the Theorem 2 of [2]. The behaviour of the remaining part is controlled by using the hypothesis of "conditional integrability" of the kernel.

Let $a \geqslant a_0(\varepsilon)$, χ_a the characteristic function of $Q(0, a)$ and

$$\tilde{G} = \tilde{G}^{(a)} = G \cdot \chi_a; \qquad H = H^{(a)} = G - \tilde{G}.$$

Obviously, $\tilde{G} \in L^1(\mathbb{R}^n)$ and

$$G_\sigma * f(t) = \tilde{G}_\sigma * f(t) + H_\sigma * f(t).$$

In every compact $K \subset \mathbb{R}^n \backslash \partial C$ it is well known that

$$|\tilde{G}_\sigma * f(t) - f(t)| < \varepsilon$$

if $\sigma < \sigma_0(\varepsilon)$. Moreover, the lemma gives

$$|H_\sigma * f(t)| < cM\varepsilon.$$

Then $G_\sigma * f - f \to 0$ uniformly with respect to σ in K.

Now we consider a neighbourhood of $t = 0$. We have

$$\begin{aligned} G_\sigma * f(t) - \chi(t)\{f(t) - f(0)\} - f(0) \int_{x_1 > t_1/\sigma} G(x)\, dx \\ = H_\sigma * f(t) + \Big\{ \tilde{G}_\sigma * f(t) - \chi(t)(f(t) - f(0)) \\ - f(0) \int_{x_1 > t_1/\sigma} \tilde{G}(x)\, dx \Big\} - f(0) \int_{x_1 > t_1/\sigma} H(x)\, dx \\ = I_1 + I_2 + I_3. \end{aligned}$$

By Lemma and (2.2)

$$|I_1| = |H_\sigma * f(t)| \leqslant cM\varepsilon,$$

where M depends only on f.

By the hypotheses on G, if $a \geqslant a_0(\varepsilon)$

$$\left| \int_{x_1 > t/\sigma} H(x)\, dx \right| < \varepsilon;$$

then

$$|I_3| \leqslant |f(0)| \cdot \varepsilon.$$

Finally, applying Theorem 2 in [2] to the function $\tilde{G}_\sigma(\int_{\mathbb{R}^n} \tilde{G}_\sigma(x)\,dx)^{-1}$ we obtain that if $a \geqslant a_1(\varepsilon, a_0)$,

$$\begin{aligned} |I_2| &= |\chi(t)(f(t)-f(0))| \cdot \left|1 - \int_{\mathbb{R}^n} \tilde{G}_\sigma(x)\,dx\right| + \eta_a(t,\sigma) \\ &\leqslant \varepsilon + \eta_a(t,\sigma), \end{aligned}$$

where $\eta_a(t,\sigma) \to 0$ when $\sigma \to 0$ uniformly with respect to t in a neighbourhood of $t=0$ in Γ_λ.

Then, if $a \geqslant \max(a_0, a_1) = \bar{a}(\varepsilon)$, we have

$$|I_1 + I_2 + I_3| \leqslant \{cM + |f(0)| + 1\}\,\varepsilon + \eta_a(t,\sigma)$$

and the theorem follows.

5. PROOF OF THEOREM 2

Theorem 2 is a consequence of the following propositions, whenever the radial function G always satisfies the hypotheses of Theorem 2.

PROPOSITION 1. *G satisfies* (2.1).

Indeed let $a_0 = a_0(\varepsilon)$ such that for every $a_0 \leqslant \rho_1 < \rho_2 < +\infty$ we have

$$\left|\int_{\rho_1}^{\rho_2} \rho^{n-1}\tilde{G}(\rho)\,d\rho\right| < \varepsilon/m_n. \tag{5.1}$$

If $S \in \Sigma$, $S \cap Q(0, a_0) = \varnothing$ we have

$$\int_S G(x)\,dx = \int_{\partial B_n} d\sigma \int_{\rho_1(\sigma)}^{\rho_2(\sigma)} \rho^{n-1}\tilde{G}(\rho)\,d\rho,$$

where $\rho_1(\sigma)$, $\rho_2(\sigma)$ are respectively the minimum and the maximum modulus of the points in the intersection of S with the ray from the origin through the point σ on the unit sphere. Then (5.1) gives $|\int_S G(x)\,dx| < \varepsilon$.

PROPOSITION 2. *If $S \in \Sigma$ and $0 \in S$*

$$\begin{aligned} \lim_{r \to +\infty} \int_{rS} G(x)\,dx &= \lim_{r \to +\infty} \int_{\|x\| < r} G(x)\,dx \\ &= m_n \int_0^{\to +\infty} \rho^{n-1}\tilde{G}(\rho)\,d\rho = 1. \end{aligned} \tag{5.2}$$

The proof follows as in Proposition 1.

PROPOSITION 3. *For every* $x_1 \in \mathbb{R}$ *there exists*

$$I(x_1) = m_{n-1} \int_0^{\to +\infty} \rho^{n-2} \tilde{G}(\sqrt{x_1^2 + \rho^2})\, d\rho \tag{5.3}$$

and $|I(x_1)| < +\infty$.

Indeed $G(x_1, \cdot)$ is a radial function $L^1_{\text{loc}}(\mathbb{R}^{n-1})$ for every $x_1 \in \mathbb{R}$. If $0 < r_0 \leqslant r < \rho$, if $\sqrt{x_1^2 + \rho^2} = t$ and $F(t) = \int_{r_0}^t u^{n-1} \tilde{G}(u)\, du$ we have

$$\begin{aligned}
\int_r^s \rho^{n-2} \tilde{G}(\sqrt{x_1^2 + \rho^2})\, d\rho &= \int_{\sqrt{r^2 + x_1^2}}^{\sqrt{s^2 + x_1^2}} t^{n-1} \frac{(t^2 - x_1^2)^{(n-3)/2}}{t^{n-2}} \tilde{G}(t)\, dt \\
&= \left[F(t) \frac{(t^2 - x_1^2)^{(n-3)/2}}{t^{n-2}} \right]_{\sqrt{r^2 + x_1^2}}^{\sqrt{s^2 + x_1^2}} \\
&\quad - \int_{\sqrt{r^2 + x_1^2}}^{\sqrt{s^2 + x_1^2}} F(t) \cdot \frac{d}{dt} \frac{(t^2 - x_1^2)^{(n-3)/2}}{t^{n-2}}\, dt;
\end{aligned}$$

then if $r \geqslant r_0(\varepsilon)$ for every $x_1 \in \mathbb{R}$,

$$\left| \int_r^s \rho^{n-2} \tilde{G}(\sqrt{x_1^2 + \rho^2})\, d\rho \right| \leqslant \frac{4\varepsilon}{r} \tag{5.4}$$

and Proposition 3 follows.

PROPOSITION 4. *For every* $x_1 \in \mathbb{R}$

$$\lim_{r \to +\infty} \int_{\substack{|x_i| \leqslant r \\ i = 2, \ldots, n}} G(x_1, \ldots, x_n)\, dx_2, \ldots, dx_n = m_{n-1} I(x_1).$$

Indeed, by Proposition 3, $G(x_1, \cdot)$ satisfies in $\mathbb{R}^{n-1}$ the hypotheses of Propositions 1 and 2.

PROPOSITION 5. $I \in L^1_{\text{loc}}(\mathbb{R})$.

Indeed for every $r > 0$ we have

$$\begin{aligned}
I(x_1) &= m_{n-1} \int_0^r \rho^{n-2} \tilde{G}(\sqrt{x_1^2 + \rho^2})\, d\rho \\
&\quad + m_{n-1} \int_r^{\to +\infty} \rho^{n-2} \tilde{G}(\sqrt{x_1^2 + \rho^2})\, d\rho \\
&= I_1^{(r)}(x_1) + I_2^{(r)}(x_1).
\end{aligned}$$

Obviously $I_1^{(r)} \in L^1_{\text{loc}}(\mathbb{R})$ for every r and $I_2^{(r)}$ is bounded by (5.4).

PROPOSITION 6. *For every $s \in \mathbb{R}$ we have*

$$\int_s^{\to +\infty} I(x_1)\, dx_1 = \int_{x_1 > s} G(x)\, dx. \tag{5.5}$$

By Proposition 1 the second term of (5.5) is well defined and finite. For every $r > s$ we have

$$\begin{aligned}
&\left| \int_s^r dx_1 \int_{\substack{|x_i| \leqslant r \\ i=2, ..., n}} G(x)\, dx_2, ..., dx_n - \int_s^r I(x_1)\, dx_1 \right| \\
&\quad \leqslant \left| \int_s^r dx_1 \left\{ \int_{\substack{|x_i| \leqslant r \\ i=2, ..., n}} G(x)\, dx_2, ..., dx_n \right.\right. \\
&\qquad \left.\left. - \int_{\|(x_2, ..., x_n)\| \leqslant r} G(x)\, dx_2, ..., dx_n \right\} \right| \\
&\qquad + \left| \int_s^r dx_1 \left\{ \int_{\|(x_2, ..., x_n)\| \leqslant r} G(x)\, dx_2, ..., dx_n - I(x_1) \right\} \right| \\
&\quad = J_1 + J_2.
\end{aligned}$$

Hence

$$J_1 = \int_{E(s, r)} \rho^{n-1} \tilde{G}(\rho)\, d\rho,$$

where $E(s, r) = \{x \in \mathbb{R}^n : s < x_1 < r,\ \|(x_2, ..., x_n)\| > r \text{ and } |(x_2, ..., x_n)| < r\}$.
If $r \geqslant r_0(\varepsilon)$ we have (as in Proposition 1)

$$|J_1| \leqslant m_n \varepsilon. \tag{5.6}$$

By (5.4)

$$\left| \int_s^v dx_1 \int_r^{\to +\infty} \rho^{n-2} \tilde{G}(\sqrt{x_1^2 + \rho^2})\, d\rho \right| \leqslant \frac{4\varepsilon}{r}(v - s).$$

Since

$$|J_2| = \left| \int_s^r dx_1 \int_r^{\to +\infty} \rho^{n-2} \tilde{G}(\sqrt{x_1^2 + \rho^2})\, d\rho \right|$$

we have $|J_2| = \leqslant (4\varepsilon/r)(r - s) < 4\varepsilon$. This inequality and (5.6) prove (5.5).

Now it is easy to see that the radial function G satisfies all hypotheses of Theorem 1. Then Theorem 2 follows by (5.5).

6. THE BOCHNER–RIESZ CASE

We recall that in $\mathbb{R}^n$, the Bochner–Riesz means of order $\alpha > 0$ of a function φ are defined via Fourier transform in the following way,

$$(S_\sigma^\alpha * \varphi)^\wedge (t) = (1 - \sigma^2 \|t\|^2)_+^\alpha \, \hat{\varphi}(t)$$

and it turns out that

$$S^\alpha(x) = S_1^\alpha(x) = \pi^{-\alpha} \Gamma(\alpha + 1) \, \|x\|^{-\alpha - (n/2)} J_{\alpha + (n/2)}(2\pi \|x\|) \qquad (6.1)$$

(see [3]) where J_β is the Bessel function of index β.

From the classical properties of Bessel functions it is easy to see that the radial function S^α satisfies the hypothesis (2.1) for $\alpha > (n-3)/2$.

In this case Theorem 2 can be reformulated in the following form (see [1] for $n = 2$).

THEOREM 3. *If the boundary of C is a smooth simple surface and if $f: \mathbb{R}^n \to \mathbb{R}$ is a function vanishing in $\mathbb{R}^n \backslash C$ and smooth enough in C and $\alpha > (n-3)/2$, then $S_\sigma^\alpha * f \to f$ as $\sigma \to 0$ uniformly in every compact subset of $\mathbb{R}^n \backslash \partial C$. Moreover*

$$S_\sigma^\alpha * f(t) = \chi(t)\{f(t) - f(0)\} + \pi^{-\alpha} \Gamma(\alpha + 1) \, m_{n-1} f(0)$$
$$\times \int_{t_1/\sigma}^{\to +\infty} x_1^{-\alpha - 1/2} J_{\alpha + 1/2}(2\pi |x_1|) \, dx_1 + \eta(t, \sigma), \qquad (6.2)$$

where $\eta(t, \sigma) \to 0$ if $\sigma \to 0$ uniformly with respect to t in a neighbourhood of $t = 0$ in Γ_λ.

Proof. If $\alpha > (n-1)/2$ this can be seen straightforwardly (see [2]). If $(n-1)/2 \geqslant \alpha > (n-3)/2$, preliminarly we observe that (2.2) is satisfied and the Theorem 2 holds. Now let us consider

$$I_1(x_1) = m_{n-1} \int_0^{\to +\infty} r^{n-2} S^\alpha(\sqrt{x_1^2 + r^2}) \, dr.$$

By (5.3), $I_1(x_1)$ is bounded and its Fourier transform is a tempered distribution.

If we prove that for every $t_1 \in \mathbb{R}$ $\hat{S}^\alpha(t_1, 0, ..., 0) = \hat{I}_1(t_1)$, the theorem will follow from (6.1). Then we have to compare $\hat{S}^\alpha$ and $\hat{I}_1$.

Let $\varphi\colon \mathbb{R} \to \mathbb{R}^+$ a spline function of order k (k sufficiently large), even, compactly supported, $\varphi(1) = 1$ if $|x| < 1/4$;

$$Z(x_1, ..., x_n) = \varphi(x_1) \cdot \varphi(\sqrt{x_2^2 + \cdots + x_n^2}) = \varphi(x_1) \cdot \varphi(r)$$

and $Z_\sigma(x) = (1/\sigma^n)\, Z(x/\sigma)$.

$$\begin{aligned}
&(\hat{S}^\alpha * \hat{Z}_\sigma)(t_1, 0, ..., 0) - (\hat{I}_1 * \hat{\varphi}_\sigma)(t_1) \\
&= \int_{\mathbb{R}} e^{it_1 x_1} \varphi(\sigma x_1) \int_{\mathbb{R}} r^{n-2} S^\alpha(\sqrt{x_1^2 + r^2})(\varphi(\sigma r) - 1)\, dr \\
&= 2\pi^{-\alpha}\, \Gamma(\alpha + 1) \int_{\mathbb{R}} e^{it_1 x_1} \varphi(\sigma x_1) \int_{r > 1/4\sigma} r^{n-2} (\sqrt{x_1^2 + r^2})^{-\alpha - n/2} \\
&\quad \times J_{\alpha + n/2}(2\pi \sqrt{x_1^2 + r^2})(\varphi(\sigma r) - 1)\, dr \\
&= 2\sqrt{2}\, \pi^{-\alpha + 1/2} \Gamma(\alpha + 1) \int_{\mathbb{R}} e^{it_1 r_1} \varphi(\sigma x_1) \int_{r > 1/4\sigma} r^{n-2} \\
&\quad \times (\sqrt{x_1^2 + r^2})^{-\alpha - ((n+1)/2)} \cos\left(\sqrt{x_1^2 + r^2} - \frac{\pi}{2}\alpha - \frac{\pi}{4}(n+1)\right) \\
&\quad \times (\varphi(\sigma r) - 1)\, dr + o(\sigma).
\end{aligned}$$

First we remark that an easy evaluation of the distance d of two consecutive zeros of $\cos(\sqrt{x_1^2 + r^2} - (\pi/2)\,\alpha - (\pi/4)(n+1)$ greater than r is

$$d \leqslant \frac{\pi^2 + 2\pi \sqrt{x_1^2 + r^2}}{2r}.$$

Since the function

$$\Phi_{\sigma, x_1}(r) = r^{n-2} (\sqrt{x_1^2 + r^2})^{-\alpha - (n+1)/2} (\varphi(\sigma r) - 1)$$

has a bounded number of zeros with respect to σ and x_1, we have

$$\begin{aligned}
&\left| \int_{r > 1/4\sigma} \Phi_{\sigma, x_1}(r) \cos\left(\sqrt{x_1^2 + r^2} - \frac{\pi}{2}\alpha - \frac{\pi}{4}(n+1)\right) dr \right| \\
&\qquad \leqslant c\sigma^{(1/2)(5 + 2\alpha - n)} \frac{\sigma\pi^2 + 2\pi \sqrt{\sigma^2 x_1^2 + 1}}{(\sqrt{\sigma^2 x_1^2 + 1})^{\alpha + ((n+1)/2)}}.
\end{aligned}$$

Then, changing variables, we obtain

$$(\hat{S}^\alpha * \hat{Z}_\sigma)(t_1, 0, ..., 0) - (\hat{I}_1 * \hat{\varphi}_\sigma)(t_1)) \leqslant c' \sigma^{(1/2)(3 + 2\alpha - n)}, \tag{6.3}$$

where c' is independent of x_1 and σ.

Because I_1 is bounded, by Lebesgue theorem it is easy to see that $\hat{I}_1 * \hat{\varphi}_\sigma \to \hat{I}_1$ in the weak*-topology of tempered distributions.

On the other hand, because $\hat{Z}$ is in L^1 we have $S^*_\alpha * \hat{Z}_\sigma \to S^*_\alpha$ uniformly in $\mathbb{R}^n$ when $\sigma \to 0$.

Then, by (6.3), $\hat{I}_1(t_1) = \hat{S}_\alpha(t_1, 0, ..., 0)$ for every $t \in \mathbb{R}$ and the theorem follows.

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